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NECESSARY AND SUFFICIENT CONDITIONS FOR HYPERPLANE TRANSVERSALS

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We prove that a finite family $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ of connected compact sets in $\mathbf{R}^d$ has a hyperplane transversal if and only if for some k there exists a set of points $P = \{p_1, p_2, \ldots, p_n\}$ (i.e., a k-dimensional labeling of the family) which spans $\mathbf{R}^k$ and every $k+2$ sets of $\mathcal{B}$ are met by a k-flat consistent with the order type of P. This is a common generalization of theorems of Hadwiger, Katchalski, Goodman-Pollack and Wenger.

Let $\mathcal{B}$ be a family of compact convex sets in $\mathbf{R}^d$. A k-transversal of $\mathcal{B}$ is a k-flat or affine subspace of dimension k which intersects each member of $\mathcal{B}$. A 1-transversal is also called a line transversal and a $d-1$-transversal in $\mathbf{R}^d$ is called a hyperplane transversal. We are interested in necessary and sufficient conditions for the existence of a hyperplane transversal of $\mathcal{B}$.

A hyperplane H separates two sets of points A and B if A lies in one of the closed half-spaces bounded by H and B lies in the other. A hyperplane H strictly separates two sets if H separates the sets and does not intersect either set. A family of sets of points in $\mathbf{R}^d$ is called *k-separable* if every j of the sets, $1 \le j \le k+1$, can be strictly separated from every $k+2-j$ sets by some hyperplane. If a family is k-separable, then there is no k-transversal for any $k+2$ sets of the family. For families of compact convex sets the converse is also true. For example, a family of compact convex sets is pairwise disjoint if and only if it is 0-separable.

By the d-order type of a numbered set of points $P = \{p_1, \ldots, p_n\} \subseteq \mathbf{R}^d$ we mean the family of orientations of its $d+1$-tuples, i.e., the family

$$\left(\operatorname{sgn}\det \begin{pmatrix} 1 & p^1_{i_0} & \cdots & p^d_{i_0} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p^1_{i_d} & \cdots & p^d_{i_d} \end{pmatrix} \right)_{1 \le i_0 < \ldots < i_d \le n}$$

(see [3] for details). This order type is *non-trivial* if the set of points affinely span $\mathbf{R}^d$. If the d-order type of P is non-trivial and $Q \subseteq P$ lies in an oriented k-flat, $k < d$, then the induced k-order type of Q is determined by the d-order type of P. A

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k-ordering of a family of sets, $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$, is defined via a corresponding set of points $P = \{p_1, p_2, \ldots, p_n\}$ in R^k. We assign as the orientation of a $k+1$-tuple of $\mathcal{B}$, say $(B_{i_0}, B_{i_1}, \ldots, B_{i_k})$, the orientation of the corresponding $k+1$-tuple of P, $(p_{i_0}, p_{i_1}, \ldots, p_{i_k})$. We say that an oriented k-flat F intersects a subset $\mathcal{A}$ of $\mathcal{B}$ *consistent* with a given k-ordering of $\mathcal{B}$ if we can choose from each $B \in \mathcal{A}$, a point in $B \cap F$, so that the chosen points have the same order type as the corresponding points in $\mathbb{R}^k$. If the family is $k-1$-separable, every choice of corresponding points generates the same order type. Thus consistency is determined by F alone.

A well known theorem of Helly states that the elements of a family $\mathcal{B}$ of compact convex sets in $\mathbb{R}^d$ meet a common point if and only if every $d+1$ elements of $\mathcal{B}$ meet a common point [9]. (For a discussion of Helly's theorem and many related theorems, see [1] and [7].) Vincensini asked whether this theorem might be generalized so that a common point is replaced by a k-flat transversal. i.e. is there a number $r(k,d), k \leq d-1$, such that the elements of a family $\mathcal{B}$ of compact convex sets in $\mathbb{R}^d$ meet a common k-flat if and only if every r elements of $\mathcal{B}$ meet a common k-flat [14]? Santaló gave an example which showed that this question has a negative answer for k positive and every r, (that $r(0,d) = d+1$ is Helly's theorem) [13]. Independently, Grünbaum and Klee showed the existence of r when the elements of $\mathcal{B}$ are in a special position (each set lying in disjoint parallel slabs) [5], [10]. In fact there is a considerable literature solving this problem for various special positions of the family $\mathcal{B}$, eg. when $\mathcal{B}$ consists of homothets or translates of a given body.

In 1957, Hadwiger removed almost all traces of special position in the case $d = 2$ by proving that given a 0-separable family of compact convex sets in the plane and a 1-ordering of the sets, if every three sets of the family are intersected by some directed line consistent with the 1-ordering then there exists a line transversal of the family [6].

In 1980, Katchalski generalized Hadwiger's result to a sufficient condition for the existence of a hyperplane transversal for a family of compact convex sets in $\mathbb{R}^d$ [10]. Katchalski showed that given a 0-separable family of compact convex sets in $\mathbb{R}^d$ and a 1-ordering of the family, if every three sets of the family are intersected by some directed line consistent with the 1-ordering then there exists a hyperplane transversal of the family.

In 1986, Goodman and Pollack gave a different generalization of Hadwiger's result to hyperplane transversals [4]. They proved that given a $d-2$-separable family of compact convex sets in $\mathbb{R}^d$ and a $d-1$-ordering of the sets, if every $d+1$ sets are intersected by some oriented hyperplane consistent with the $d-1$ ordering then there exists a hyperplane transversal of the family.

Both Katchalski's result and Goodman and Pollack's result are generalizations of Hadwiger's theorem. However, neither of these generalizations contains the other and it was not clear how they were related. The relationship between these theorems is made by the following theorem: Given a $k-1$-separable family of compact convex sets in $\mathbb{R}^d$ and a k-ordering of the family, if every $k+2$ sets are intersected by some oriented k-flat consistent with the k-ordering then there exists a hyperplane transversal of the family.

When $k = 1$, $d = 2$ we have Hadwiger's theorem, $k = 1$ is Katchalski's theorem and $k = d-1$ is the theorem of Goodman and Pollack. A proof of this theorem is possible using methods from all of these papers. Instead of proving this theorem we shall prove a generalization which makes no hypothesis of separability and applies equally well to families of connected sets. That the history of this problem was about families of convex sets has diverted attention from the proper setting for this theorem and the appropriate conditions for this theorem. The only relevant connection between connected sets and convex sets is that the projection of a connected set on a line is convex. The method of proof is derived from Wenger [15], who in 1987 removed all trace of special position from Hadwiger's theorem by proving that a family of compact convex sets in $\mathbf{R}^2$ has a line transversal if and only if there is a 1-ordering of the family such that every 3 sets of the family are intersected by some oriented line consistent with the 1-ordering.

Theorem 1. *A family of compact connected sets has a hyperplane transversal if and only if for some k, $0 \leq k \leq d-1$, there is a non-trivial k-ordering of the family, such that every $k+2$ sets are intersected by some oriented k-flat consistent with the k-ordering.*

It suffices to prove Theorem 1 for a finite family, for the Theorem then follows by a standard compactness argument. The proof proceeds as follows. We define a continuous anti-symmetric ($f(-v) = -f(v)$) function $f : S^{d-1} \to \mathbf{R}^k$ which by the Borsuk-Ulam theorem (see [12]) has a zero. The existence of this zero implies the existence of a transversal or that two subsets of the k-ordering have intersecting convex hulls which is impossible because the corresponding sets of the family are separated by a hyperplane. We need the following lemma which is a slight generalization of Lemma 1 of [4].

Lemma 2. *Let $P = \{p_1, \ldots, p_n\}$ and $P' = \{p'_1, \ldots, p'_n\}$ be corresponding sets of n labeled points in $\mathbf{R}^k$ which have the same non-trivial order type in $\mathbf{R}^k$. The disjoint subsets U and V of P have non-disjoint convex hulls (i.e. (U, V) is a Radon partition of $U \cup V$ or (U, V) is a Radon partition in P) if and only if the corresponding subsets, U' and V' of P' do.*

Proof. Without loss of generality we may assume that U and V are minimal with this property, i.e. there do not exist subsets U_1 of U and V_1 of V whose convex hulls intersect. Hence, by the Hare-Kenelly theorem (see [2] or [8]), S consists of $j+2$ points in general position where $j = \dim(\text{aff}(S))$ and by [4] this partition is unique. Thus U, V is a minimal Radon partition (see [4] or [2] where it is called a primitive Radon partition) of $S = U \cup V$. The lemma now follows from the fact that the minimal Radon partition of S is determined by the j-order type S (see [4]) which is in turn determined by the k order type of P, since it is non-trivial. ∎

Proof of Theorem 1. Let $P = \{p_1, \ldots, p_n\}$ in $\mathbf{R}^k$, be the point set corresponding to $\mathcal{B}$ which defines a non-trivial k-ordering of $\mathcal{B}$ so that every $k+2$ sets of $\mathcal{B}$ are intersected by an orientated k-flat consistent with this k-ordering. For $v \in S^{d-1}$ and a hyperplane H orthogonal to v, denote by H^+ the closed half-space bounded by H. Let $H_l(v)$ be the unique hyperplane orthogonal to v such that every $b \in \mathcal{B}$ meets $H_l^+(v)$ and some $b \in \mathcal{B}$ is a subset of $H_l^-(v)$. We let $H_r(v) = H_l(-v)$. Finally we let $H(v)$ be the hyperplane orthogonal to v which is half-way between $H_l(v)$ and $H_r(v)$. Hence, with $d(b, H)$ denoting the distance from the compact set b to the hyperplane

H, the function $f : S^{d-1} \to \mathbf{R}^k$ defined by

$$f(v) = \sum (p_i - p_j) \min_{\substack{b_i \leq H^+(v) \\ b_j \leq H^-(v)}} \{d(b_i, H(v)), d(b_j, H(v))\}$$

is continuous and $f(-v) = -f(v)$. Hence, by the Borsuk-Ulam theorem (see [12]), there is a v such that $f(v) = 0$. That $f(v) = 0$ means that either $H(v)$ is a transversal for $\mathcal{B}$ and $\min\{d(b_i, H(v)), d(b_j, H(v))\} = 0$ for all i, j or some convex combination of $p_i - p_j$ equals zero. (Note that if $H(v)$ missed some set in $\mathcal{B}$ it would have to separate two sets in $\mathcal{B}$ by construction.) We shall prove that a convex combination of the vectors $p_i - p_j$ cannot equal zero. Suppose that some convex combination does equal zero. Then P has two subsets S, T whose convex hulls intersect where, $S = \{p_i | b_i \subset H^+(v)\}$ and $T = \{p_i | b_i \subset H^-(v)\}$. Hence, by Kirchberger's theorem (see [1] or [11]), there is a $k+2$ element subset U of $S \cup T$ such that $conv(S \cap U) \cap conv(T \cap U) \neq \emptyset$. The corresponding subset of $\mathcal{B}$, say U', is met by a hyperplane H consistent with the order-type of U. By Lemma 1 this implies that the points from the sets S', T' corresponding to $S \cap U$, $T \cap U$ form a Radon partition and therefore that the convex hull S' meets the convex hull of T'. But this is impossible since the sets S', T' corresponding to S, T are separated by the hyperplane $H(v)$. We conclude that $H(v)$ is a hyperplane transversal for $\mathcal{B}$. ∎

Remarks. In a natural sense theorem 1 settles Vincensini's problem for the case of hyperplane transversals. It would be very interesting to discover what sort of conditions would be appropriate for k-flat transversals. We know of no reasonable sufficient conditions even in the simplest case, of line transversals in three dimensions.

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